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Multiple-valued extensions of analogical proportions

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Abstract

This paper provides a discussion of different multiple-valued extensions of logical expressions of analogical proportions that are equivalent in the Boolean case. It is advocated that these extensions may serve different goals, or at least reflect different views. Boolean expressions of analogical proportions, modeling statements of the form A is to B as C is to D, are first restated and commented. An approach to handling the case of binary attributes that may not apply in the Boolean setting is also outlined via a proper encoding.

Keywords: Multiple-valued logic; Analogical proportion; Similarity; Dissimilarity; Interpolation

1. Introduction

Comparing situations, a core step in reasoning processes, is often a matter of assessing similarities and dissimilarities, which may be a matter of degree. This has motivated fuzzy logic-based works in case-based reasoning [7,9] but also in approximate reasoning [27], although with different approaches. The works by Francesc Esteva are in part dedicated to this kind of topic. In particular, some were published in joint papers with the two first authors of this note on fuzzy case-based reasoning [5], and on interpolative reasoning [6,10]. More recently, the two last authors of this note have devoted a significant effort to the development of a formal logic approach to analogical reasoning [12,15].

Analogical proportions are statements of the form A is to B as C is to D, often denoted $A : B :: C : D$, where the comparison made between situations A and B is balanced against the comparison made between situations C and D. One may think of these situations as described by vectors of Boolean or graded attribute values. This kind of pattern for analogy has been already implicitly at work in fuzzy set-based approximate reasoning [24,23,3] and is also at the root of case-based reasoning [8]. In the numerical case, the analogy principle is typically rendered by the equality $\frac{a}{b} = \frac{c}{d}$. However, it is only recently that a rigorous notion of *logical* proportion, leading to a sound logical formula for $A : B :: C : D$ in terms of Boolean attributes has been proposed (see [16] for an overview).

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Table 1
The six Boolean patterns for which an analogical proportion is true.

0	0	0	0
1	1	1	1
0	0	1	1
1	1	0	0
0	1	0	1
1	0	1	0

The aim of this note is to review the logical approach to analogical reasoning developed by the second and third authors, and to discuss the potential merits of many-valued extensions of several logical expressions of analogical proportions that are equivalent in the Boolean case. These propositional logic expressions are first recalled in Section 2. Section 3 discusses the handling of the case of binary attributes, in the case when they may not apply to all situations. Then Section 4 is devoted to two different many-valued extensions of the Boolean analogical proportions, when attribute values are assumed to be precisely known.

2. The Boolean expressions of analogical proportion

Let us consider the case where situations or objects A, B, C, and D are described in terms of binary attributes (i.e. that are either true or false). Let us focus on a particular Boolean attribute or predicate $\mathcal{X}$ of interest, and let a, b, c, d (the values of) the respective Boolean variables pertaining to this attribute in each of the four situations A, B, C, and D that is, $a = \mathcal{X}(A)$, $b = \mathcal{X}(B)$, $c = \mathcal{X}(C)$, $d = \mathcal{X}(D)$. In the following, a Boolean variable, or its value, are denoted in the same way, as far as this abuse of notation does not create any confusion.

2.1. Properties of the analogical proportion

In the classical logical setting, an analogical proportion can be viewed as a quaternary connective linking four Boolean variables a, b, c, and d, which will be denoted $a : b :: c : d$, as first proposed in [11,12]. Then, an analogical proportion holds between four situations A, B, C, and D, i.e. $A : B :: C : D$, if and only if the analogical proportion $a : b :: c : d$ holds true for each of the binary attributes used for describing the situations.

The following properties describe the logical behavior of the analogical proportion [16]:

1. $a : b :: a : b$ (reflexivity)
2. $a : b :: c : d \equiv c : d :: a : b$ (symmetry)
3. $a : b :: c : d \equiv a : c :: b : d$ (central permutation)

Although analogical proportions had not been thought of in a logical way until recently, these properties have been always associated to the idea of analogical proportion since Aristotle time [4,16], as these properties are true for numerical ratios. Thus, it is natural that these properties should hold for $a : b :: c : d$ as well.

Thus in particular, by reflexivity and central permutation, $a : b :: a : b$ and $a : a :: b : b$ should be tautologies. This means that in the truth table of $a : b :: c : d$, the 6 patterns given in Table 1 makes it true. Moreover, $a : b :: c : d$ should be false for the 10 other quaternary patterns (among the $2^4 = 16$ possible ones). These 10 other patterns are of two kinds:

- $(1, 0, 0, 1)$ and $(0, 1, 1, 0)$ that are certainly undesirable since accepting them as making the analogical proportion true would lead to admit that $A : B :: C : D$ entails $B : A :: C : D$, which sounds strange;
- the 8 patterns are those with an odd number of '1's or '0's such as $(1, 0, 0, 0)$ or $(0, 1, 1, 1)$. They express the idea that one among a, b, c, d is different from the three others, which has nothing to do with the statement a is to b as c is to d.

It can be easily checked that beyond reflexivity, symmetry and central permutation, the following properties hold for $a : b :: c : d$ defined by truth table on Table 1:

- $a : b :: c : d \equiv d : b :: c : a$
- $a : b :: c : d \equiv \neg a : \neg b :: \neg c : \neg d$

The first equivalence is the permutation of the extremes and derives from symmetry and central permutation in any setting. The second property, derivable from the three basic properties in the Boolean setting [16], is also important since it expresses some invariance with respect to encoding (it is the same to describe a situation in terms of a property or of its negation).

As can be easily seen on the patterns of Table 1, when the analogical proportion $a : b :: c : d$ holds true, either the truth pattern for (a, c) is the same as the one for (b, d) (4 patterns among 6), or the truth change from a to b goes along with a change in the same direction (from true to false, or from false to true) from c to d. In particular, $a : b :: c : d$ is false for the patterns $(1, 0, 0, 1)$ and $(0, 1, 1, 0)$; it is also false for the 8 patterns with one '1' or one '0' such as $(1, 0, 0, 0)$ or $(0, 1, 1, 1)$.

2.2. The logical expressions for analogy

A logical expression corresponding to the truth Table 1 has been first proposed in [11,12]. It defines what $a : b :: c : d$ logically means:

$$a : b :: c : d = (a \wedge \neg b \equiv c \wedge \neg d) \wedge (\neg a \wedge b \equiv \neg c \wedge d) \tag{1}$$

Then, $A : B :: C : D$ means that A differs from B as C differs from D on the considered attribute. More precisely, $a \wedge \neg b$ (resp. $\neg a \wedge b$) computes the difference or dissimilarity between a and b (resp. b and a), which is equated to the difference between c and d (resp. d and c). This logical view has been further studied and discussed in [16].

Note also that by changing a into $\neg a$, b into $\neg c$, c into d, d into b in the logical expression (1), we get the logical expression of $a : c :: b : d$, (which is equivalent to $a : b :: c : d$ using central permutation). We can infer that if a is to c as b is to d, then *not* a is to *not* c what d is to b.

The central and extreme permutation properties are easy to see on the truth table, while they are not obvious on Eq. (1) (while symmetry is patent). In particular, these permutation properties do not hold separately on each of the two sides of the external conjunction, i.e. $a \wedge \neg b \equiv c \wedge \neg d$ and $a \wedge \neg c \equiv b \wedge \neg d$ do not have the same truth tables.

Finally, the above expression of the logical proportion can be equivalently written using material implication and conjunction only as [11,12]:

$$a : b :: c : d = (a \to b \equiv c \to d) \wedge (b \to a \equiv d \to c), \tag{2}$$

which means that we can reason from c to d and back, just as we do from a to b and back. Note that the truth of $(a \to b) \to (c \to d)$ comes down to the license to deduce d from the rule $a \to b$ and the fact c (this is the principle of the so-called "triple I method" [25]). Under this view, the logical expression of analogy looks stronger than usual inference, since it comes down to the above license to deduce d from the rule $a \to b$ and the fact c, plus three other inferences (of b from $c \to d$ and a, and likewise of a and c using the right-hand side of the pattern (2)).

There exists another remarkable, equivalent expression of the analogical proportion [12] where the permutation properties are obvious (as well as symmetry). Namely

$$a : b :: c : d = (a \wedge d \equiv b \wedge c) \ \wedge \ (\neg a \wedge \neg d \equiv \neg b \wedge \neg c) \tag{3}$$

As can be seen, this expression focuses on similarities, while expression (1) focuses on dissimilarities. Indeed $a \wedge d$ and $\neg a \wedge \neg d$ are positive and negative similarity indicators respectively [16]. Thus, Eq. (3) reads A is to B as C is to D if and only if A and D are similar just as B and C are (on the considered attribute). This is not so obvious a translation of the idea of analogy since the latter usually expresses that A compares to B just as C compares to D, an idea better captured by expression (1). Moreover it is straightforward to see that Eq. (3) can be rewritten equivalently without negation but using disjunction:

$$a : b :: c : d = (a \wedge d \equiv b \wedge c) \ \wedge \ (a \vee d \equiv b \vee c) \tag{4}$$

Interestingly enough, this expression was precisely considered by Piaget [14] as defining the idea of *logical proportion* (applicable not only on atoms, but on any 4-tuple of propositional formulas). He viewed it as a counterpart of the idea of numerical *geometric*[1] proportions (since $\frac{a}{b} = \frac{c}{d}$ if and only if $a \times d = b \times c$), although he apparently never made any connection with the idea of analogical proportion.

Remark 1. Recently, an expression of the analogical proportion as a *disjunction* of two conjunctions of equivalences has been also noticed [13]:

$$a : b :: c : d = ((a \equiv b) \wedge (c \equiv d)) \vee ((a \equiv c) \wedge (b \equiv d)). \tag{5}$$

This expression directly compares a, b, c, d and does not use dissimilarity or similarity indicators as in (3) (or (4)). It may seem less appropriate with respect to the intended meaning of an analogical proportion. However, reflexivity, symmetry, central permutation, and invariance with respect to encoding are easy to check on (5).

2.3. A set-theoretic view

The apparent discrepancy between the expressions (1) and (3) of the analogical proportion $a : b :: c : d$ can be better understood through a set-theoretic, equivalent view of the analogical proportion, due to [22]. As already said, $a : b :: c : d$ can be true only for the six patterns of Table 1. Consider the set-theoretic counterpart of (1), that is, interpret A, B, C, D as the subsets of models of the Boolean variables a, b, c, d respectively. Then, (1) reads:

$$(A \setminus B = C \setminus D) \text{ and } (B \setminus A = D \setminus C)$$

Introduce the sets

- $X = A \setminus B = C \setminus D, \quad Y = B \setminus A = D \setminus C;$
- $U = A \cap B \cap C \cap D$, $Z = (A \cap B) \setminus (C \cap D)$, $T = (C \cap D) \setminus (A \cap B)$, and finally $V = \overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{D}$.

Then it is clear that X, Y, Z, Y, U, V form a partition of the whole set of interpretations of the language, Ω, and

$$A = X \cup Z \cup U, \quad B = Y \cup Z \cup U, \quad C = X \cup T \cup U, \quad D = Y \cup T \cup U.$$

Then it is easy to check that under these constraints, the set-theoretic-counterpart of (3) holds, that is:

$$A \cap D = B \cap C = U \text{ and } \overline{A} \cap \overline{D} = \overline{B} \cap \overline{C} = V. \tag{6}$$

Hence, $A \cup D = X \cup Y \cup Z \cup T \cup U = B \cup C$, which lays bare the link between expressions (1) and (4).

The setting in [22] is in fact the one of a Boolean table with objects A, B, C, D, etc., in lines, and properties $\mathcal{X}_i$, $i = 1, \dots, n$, in columns. And A, B, C, D are then understood as the subsets of Boolean properties that hold true for each of the four respective objects (or situations). So letting $a_i = \mathcal{X}_i(A)$, etc. The analogy pattern $A : B :: C : D$ stands for $a_i : b_i :: c_i : d_i$, $i = 1, \dots, n$, which can be denoted by $(a_1, \dots, a_n) : (b_1, \dots, b_n) :: (c_1, \dots, c_n) : (d_1, \dots, d_n)$.

Hence, for instance, Z (resp. T) is the set of properties that hold both in situations A and B (resp. C and D), and do not in situations C and D (resp. A and B). The last expression (6) clearly means that the properties common to A and D are those common to B and C, and likewise for their missing properties.

These different views of the analogical proportion are all equivalent in the Boolean setting. It is no longer the case when we consider their possible extensions for handling numerical attributes, as explained in the rest of this paper, where we provide a comparative discussion of the merits of the two main existing options, based on previous results [15,17]. However, we first revisit the case of attributes that may not be applicable to some situations.

3. Beyond Boolean analogical proportions: attributes that may fail to apply

Binary attributes may fail to be applicable to all objects or situations. In this situation, some propositions are neither true nor false. This is captured by some kind of null-values in databases for instance. A multiple-valued logic modeling has also been recently proposed to handle the adaptation of analogy patterns to this case [17]. It is indeed tempting to introduce a third truth value meaning “not applicable” (denoted by *na* for short), leading to a truth table with $3^4 = 81$

[1] This is true as well for *arithmetic* proportions of the form $a - b = c - d$, which is equivalent to $a - c = b - d$, and to $a + d = b + c$ [16].

lines. Then it may seem to be intuitively appealing to acknowledge that all patterns of the form (s, s, s, s), (s, s, t, t) and (s, t, s, t) where $s, t \in \{0, 1, na\}$ satisfy $a : b :: c : d$. It consists of 15 patterns:

$$(1, 1, 1, 1), (0, 0, 0, 0), (na, na, na, na),$$
$$(1, 0, 1, 0), (0, 1, 0, 1), (1, na, 1, na), (na, 1, na, 1), (0, na, 0, na), (na, 0, na, 0),$$
$$(1, 1, 0, 0), (0, 0, 1, 1), (1, 1, na, na), (na, na, 1, 1), (na, na, 0, 0), (0, 0, na, na),$$

while all the $81 - 15 = 66$ other patterns built from the set $\{na, 0, 1\}$ will be considered as making $a : b :: c : d$ false.

Extensions of definitions (1) or (3) with suitable connectives for making $a : b :: c : d$ true for the 15 desired patterns, and false otherwise, have been proposed in [17]. Some of these tri-valued connectives were quite unusual. Namely, two different three-valued extensions of logical difference were needed in (1) or equivalently two circular negations (in the style of Post logic) and an associative nonmonotonic conjunction already studied by Walker [26] for combining three-valued conditional events (that are not applicable when their antecedent is false).

In the following, we indicate a simpler method for obtaining the same partition of the patterns into "true" and "false" categories, i.e. remaining in a Boolean setting. This view comes down to using a non-compact encoding using two attributes instead of one, and considering na as a kind of additional Boolean attribute. Namely, rather than asking if an attribute $\mathcal{X}$ is true in a considered situation a, or false or yet not applicable, we shall view the applicability of attribute $\mathcal{X}$ itself as a binary attribute $\mathcal{Y}_1 = appl(\mathcal{X})$ which can be only true or false, and a second attribute $\mathcal{Y}_2$ defined as the conjunction $appl(\mathcal{X}) \wedge \mathcal{X}$, which is true if $\mathcal{X}$ is applicable and true, and false otherwise. Thus, the values 1, 0, and na will be encoded by the pairs $(1, 1)$, $(1, 0)$, and $(0, 0)$ respectively, corresponding to the two Boolean attributes $\mathcal{Y}_1, \mathcal{Y}_2$. In fact, this may be viewed as a special case of a general method for dealing with discrete domains having more than two values, using Boolean variables.

Proposition 1. *All forms of the analogical proportion are valid for the three patterns* (s, s, s, s), (s, s, t, t) *and* (s, t, s, t)*, where* $s, t \in \{0, 1, na\}$*, on both components of the two-variable encodings of possibly non-applicable attribute, and only those forms.*

Proof. The validity of the patterns (s, s, s, s), (s, s, t, t), (s, t, s, t) is easy to check. For example, for (s, t, s, t), Boolean 4-tuples obtained from components of the vectors $(\mathcal{Y}_1, \mathcal{Y}_2)$ will be either equal or will form the pattern (s, t, s, t). For instance, the patterns $(0, na, 0, na)$ supposed to satisfy the analogical proportion expression will then be encoded as an analogical proportion between pairs in $\mathbb{B}^2$:

$$(1, 0) : (0, 0) :: (1, 0) : (0, 0),$$

which clearly holds in a componentwise manner, since the 2 Boolean 4-tuples $(1, 0, 1, 0)$ and $(0, 0, 0, 0)$ are valid patterns in the truth table of $a : b :: c : d$. Conversely, any non-valid patterns among the 66 ones, falsifies the analogical proportion expression for the two components in the binary encoding. It concerns patterns (s, t, t, s), e.g. $(1, na, na, 1)$, that include at least one similar Boolean 4-tuple, hence not valid. Patterns (s, t, t, t) and their permutations, having three identical components, will contain one Boolean 4-tuple with an odd number of ones and zeros, hence not valid. Finally, consider patterns with three distinct components (s, t, u, t), such as $(0, 1, na, 1)$, and their permutations. Noticing that Boolean triples extracted from (s, t, u) (with s, t, u distinct) contain one 0 and two 0's respectively for attributes $\mathcal{Y}_1$, and $\mathcal{Y}_2$, (s, t, u, t) will contain at least one Boolean 4-tuple with an odd number of 1's, hence not valid. □

As an example, consider a set of individuals, and attribute $\mathcal{X}$ indicating pregnancy. Clearly, this is a partially applicable attribute, depending on the sex. So, $\mathcal{Y}_1 = appl(\mathcal{X})$ indicates the sex of individuals, and takes value 1 for women. Then the validity of analogical proportions after the above approach is clear: for instance, (na, na, na, na) corresponds to four men, that are clearly analogous with respect to pregnancy, $(na, na, 1, 1)$, means that a man is to a man what a pregnant woman is to a pregnant woman, $(0, na, 0, na)$ means that a non-pregnant woman is to a man what another non-pregnant woman is to another man, etc.

Thus, the case of non-applicable Boolean attributes may still be handled in a satisfactory way in a Boolean setting, insofar as the non-applicability of one attribute can be interpreted as the truth of another meaningful property.

Remark 2. In the above approach, the third truth-value is viewed as a supplementary Boolean attribute, which explains why the result is always true or false. Yet, one might object to the assumption that a pattern such as *(na, na, na, na)* can still stand as an analogical proportion. Taking the idea of non-applicable for granted, one may consider that in this case the whole pattern is not applicable to four objects with non-applicable properties, that is we may look for connectives that take the value *na* in some situations (after all, why should we see a pattern of analogical proportion between attributes on situations where these attributes do not apply?). There are two alternative ways of handling the latter idea of non-applicable in a three-valued setting. One may assign the value *na* to $a : b :: c : d$ in all cases when the value *na* appears, in the spirit of strong truth-tables of Kleene three-valued logics. Such contamination effect would not lead to a very useful framework (it would lead us to restrict to the 6 Boolean valid patterns and drop the non-Boolean ones). The other approach leads to considering *na* as a semi-group identity in the truth tables of conjunction and disjunction, in the style of Sobocinski's logic [21]; conjunction and disjunction are uninorms with respective absorbing elements 0 and 1 [2]. Then the truth-value *na* abstains from intervening whenever possible. Only the pattern *(na, na, na, na)* has truth-value *na*. However, using Sobocinski's connectives on analogy pattern (5), we obtain a result such as 1: *na* :: 1 : 1 = 1, which looks highly questionable. So, finding an alternative three-valued logic approach (simpler than the one outlined in [17]), where the logical proportion never takes the value *na*, is an open problem.

4. Gradual extensions of the analogical proportion

When the attributes used for describing situations are numerical, the logical definition of analogical proportion needs to be extended. For simplicity, as in the Boolean case, we assume that all numerical attribute domains are then (proportionally) rescaled on the unit interval [0, 1].

In contrast with the case of non-applicable values discussed above, we now expect that the multiple-valued extension of $a : b :: c : d$ from $\mathbb{B}^4$ to $[0, 1]^4$ takes values intermediary between 0 and 1 (e.g., $a : b :: c : d$ for the pattern $(0.9, 0, 1, 0)$ can be neither1 nor 0, but should rather have a high value since 0.9 is close to 1). For simplicity, in the sequel, the symbol a denotes both a variable and its truth-value.

The multiple-valued logic extension of the equivalent logical expressions of analogical proportion encountered in Section 2 first requires the choice of connectives for the external conjunction and the two equivalences. The following choices are of special interest [15,17]:

i) the central, external conjunction $\wedge$ is taken as equal to the minimum. It is desirable here that the result of the conjunction be non-zero if both conjuncts have positive truth-values; in this case, nilpotent conjunctions are excluded. Finally, there is an argument of simplicity and one of dependence between terms that preclude the use of product.
ii) $s \equiv t$ is taken as $\min(s \rightarrow_L t, t \rightarrow_L s)$ where $\rightarrow_L$ is Łukasiewicz implication, defined by $s \rightarrow_L t = \min(1, 1 - s + t)$, for $\mathcal{L} = [0, 1]$ and thus $s \equiv t = 1 - |s - t|$. There are two arguments in favor of this choice. First, with this definition, $s \equiv t$ takes the truth value 1 if and only if $s = t$, which is fully in the spirit of exact comparisons; moreover this index is explicitly related to the usual distance between numerical values (expressed by the absolute value of the difference between s and t).

Remark 3. The minimal algebraic structures needed to extend the equivalent logical proportion patterns presented in Section 2 to the multiple-valued case are not the same. For instance, the expressions (2), (4) and (5) only need a residuated lattice (e.g. min, max and Gödel implication) to be extended. However, one expects to also use an involutive negation in (1) and (3).

4.1. A conservative graded extension

A straightforward extension of expression (4) is obtained by also taking minimum for the internal conjunction and maximum for the internal disjunction [17]. It yields:

$$a : b ::_C c : d = \min(1 - |\max(a, d) - \max(b, c)|, 1 - |\min(a, d) - \min(b, c)|) \tag{7}$$

It is an extension of expression (3) as well, using $1 - \cdot$ for negation. It can be easily seen that reflexivity, symmetry, central permutation, and invariance with respect to encoding continue to hold. Then it is clear that:

Proposition 2. *$a : b ::_C c : d = 1$ if and only if $\min(a, d) = \min(b, c)$ and $\max(a, d) = \max(b, c)$.*

It means that only patterns of the form (x, y, x, y) or (x, x, y, y) where $x, y \in [0, 1]$ and possibly $x = y$ make $a : b ::_C c : d$ fully true, generalizing the six cases of Table 1 and replacing 0 and 1 by x and y. In these patterns, a, b, c, d take values on a binary set $\{x, y\} \subset [0, 1]$ only, and the degree of change from a to b, if any, must be exactly the same as the one from c to d, in *direction* (the first requirement enforcing the same amplitude of change). This is clearly a conservative view of graded analogy that remains close in spirit to the Boolean case.

In contrast, for instance, $0.2 : 0.4 ::_C 0.6 : 0.8 = 0.2 : 0.4 ::_C 0.4 : 0.6 = 0.8$, rather than 1 since in both cases there are more than two distinct truth-values for a, b, c, d, despite the fact that the changes from a to b and from c to d have the same direction and amplitude. These cases are of the form $(x, x + z, y, y + z)$ with $x + z \le 1$, $y + z \le 1$, for which we have:

$$x : (x + z) ::_C y : (y + z) = \begin{cases} 1 - z \text{ if } \min(x + z, y + z) \le \max(x, y) \\ 1 - |x - y| \text{ otherwise}, \end{cases} \tag{8}$$

or equivalently, $x : (x + z) ::_C y : (y + z) = 1 - \min(z, |x - y|)$. It lays bare the role played by the difference between the truth-values of a and b, a and c, c and d. If $x + z = y$ (three distinct values, $x, y, 2y - x \le 1$), then $x : y ::_C y : (2y - x) = 1 - |x - y| = 1 - z$.

Besides, it is obvious as well from the expression (7) that:

Proposition 3. *$a : b ::_C c : d = 0$ if and only if $|\min(a, d) - \min(b, c)| = 1$ or $|\max(a, d) - \max(b, c)| = 1$.*

In other words, the only patterns fully falsifying the analogical proportion are of the form $1 : 0 ::_C x : 1$ or $0 : 1 ::_C x : 0, \forall x \in [0, 1]$ (and the other patterns obtained from these two by symmetry and central permutation). Thus, $a : b ::_C c : d = 0$ if and only if there is a maximal difference between a and b, while a and d are equal, whatever the remaining term (and the like for the ones obtained by symmetry and central permutation). Interestingly, the two extreme patterns of the form $1 : 0 ::_C x : 1$ are $1 : 0 :: 0 : 1$ and $1 : 0 :: 1 : 1$ that have truth-value 0, which explains the same holds if $0 < x < 1$. Again this is a conservative extension of the false Boolean cases.

If we use a 3-level scale $\mathcal{L} = \{\alpha_0 = 0, \alpha_1 = 1/2, \alpha_2 = 1\}$, $a : b ::_C c : d$ defined by (7) is equal to 1 exactly for the same 15 patterns (replacing *na* by 1/2) where truth was preserved in the case of non-applicable values. It is equal to 0 for the 18 patterns of the form $1 : 0 ::_C x : 1$ or $0 : 1 ::_C x : 0$ (and the ones obtained by symmetry or central permutation). In the $3^4 - (15 + 18) = 48$ other cases, $a : b ::_C c : d = 1/2$.

Remark 4. Note that if we use product instead of minimum for the inner conjunctions, and the probabilistic sum $(x + y - xy)$ in expression (4), then $a : b ::_C c : d = 1$ if and only if both equalities $\frac{a}{b} = \frac{c}{d}$ and $a - b = c - d$ hold, that is, both arithmetic and geometric proportions must hold, which yields $a = c$ and $b = d$ or $a = b$ and $c = d$. It generalizes again the same six cases of Table 1 replacing 0 and 1 by a and b, respectively, as with minimum and maximum. Using Łukasiewicz conjunction and disjunction instead would only yield the condition $a - b = c - d$.

The alternative disjunctive expression (5) could be also considered as a starting point for the extension of $a : b ::_C c : d$ to the multiple-valued case. Namely, using min for $\wedge$, max for $\vee$, and $1 - |\cdot - \cdot|$ for $\equiv$, we obtain (factoring out the negation appearing in the many-valued equivalences):

$$a : b ::_C c : d = 1 - \min(\max(|a - b|, |c - d|), \max(|a - c|, |b - d|)) \tag{9}$$

Proposition 4. *Numerical expressions (9) and (7) coincide.*

Proof. Indeed, it can be seen that using (7) $a : b ::_C c : d$ can be also written as:

$$\begin{cases} 1 - \max(|a - b|, |c - d|) & \text{if } a \ge d \text{ and } b \ge c, \text{ or } a \le d \text{ and } b \le c \\ 1 - \max(|a - c|, |b - d|) & \text{if } a \ge d \text{ and } c \ge b, \text{ or } a \le d \text{ and } c \le b \end{cases} \tag{10}$$

Besides, we have $\max(|a - b|, |c - d|) \le \max(|a - c|, |b - d|)$ if $a \ge d$ and $b \ge c$. This can be easily checked by letting $a = d + x$ and $b = c + y$ with $x > 0$ and $y > 0$ and rewriting the inequality as $\max(|d - b + x|, |d - b + y|) \le$

$\max((|d-b+x+y|, |d-b|)$. The three other cases are similar. This shows the equality of (9) and (10) (and thus (7)). $\square$

Remark 5. However, (7) no longer coincides with the extensions of two other disjunctive expressions of the logical proportion, involving negations of equivalences in place of equivalences in one of the two conjunctions (and suitable permutations of a, b, c, d) [13]:

$$\begin{aligned} a : b :: c : d &= ((a \not\equiv d) \wedge (b \not\equiv c)) \vee ((a \equiv b) \wedge (c \equiv d)) \\ &= ((a \not\equiv d) \wedge (b \not\equiv c)) \vee ((a \equiv c) \wedge (b \equiv d)) \end{aligned}$$

whose respective multiple-valued extensions

$$a : b :: c : d = \max(\min(|a-d|, |b-c|), \min(1-|a-b|, 1-|c-d|))$$

$$a : b :: c : d = \max(\min(|a-d|, |b-c|), \min(1-|a-c|, 1-|b-d|))$$

are such that $1/2 : 1/2 :: 1 : 1 = 1$ and $1/2 : 1 :: 1/2 : 1 = 1/2$ (resp. $1/2 : 1/2 :: 1 : 1 = 1/2$ and $1/2 : 1 :: 1/2 : 1 = 1$), i.e. central permutation does not hold.

4.2. A liberal graded extension

The multiple-valued extension of (4) (or (5)) has truth value 1 only for patterns of the form (x, y, x, y) or (x, x, y, y) where $x, y \in \mathcal{L}$ and possibly $x = y$. It makes sense on ordinal lattice-like truth scales [1], but it sounds like a very restrictive view of graded analogy, not in line with its usual numerical forms. This will be no longer the case with the multiple-valued extension of expression (1), which is now discussed. Keeping the same choices of connectives for central conjunction and equivalence, we interpret $s \wedge \neg t$ in (1) as a bounded difference $\max(0, s-t) = 1 - (s \to_L t)$, in other words, this is like using Łukasiewicz implication in expression (2). The resulting expression is then

$$a : b ::_L c : d = \begin{cases} 1 - |(a-b) - (c-d)|, \\ \quad \text{if } a \geq b \text{ and } c \geq d, \text{ or } a \leq b \text{ and } c \leq d \\ 1 - \max(|a-b|, |c-d|), \\ \quad \text{if } a \leq b \text{ and } c \geq d, \text{ or } a \geq b \text{ and } c \leq d \end{cases} \tag{11}$$

The expression (11) still satisfies reflexivity, symmetry, central permutation, and invariance with respect to encoding.

It can be checked that the analogical proportion is valid when the changes from a to b and from c to d have the same direction and amplitude, and there is no longer any requirement on the number of distinct values a, b, c, d:

Proposition 5. *$a : b ::_L c : d = 1$ if and only if $a - b = c - d$.*

For instance, $0.2 : 0.4 ::_L 0.6 : 0.8 = 0.2 : 0.4 ::_L 0.4 : 0.6 = 1$, in contrast with the conservative approach of the previous section.

In case of a three-valued truth scale $\mathcal{L} = \{0, 1/2, 1\}$, there are 4 more patterns that satisfy $a : b ::_L c : d = 1$ with respect to (7), namely $(0, 1/2, 1/2, 1)$, $(1, 1/2, 1/2, 0)$, $(1/2, 1, 0, 1/2)$, and $(1/2, 0, 1, 1/2)$, on top of the 15 patterns for which (7) was fully true and are still true here. So this is a more liberal view of graded analogy, more precisely one of the known natural numerical expressions of it, in terms of equal differences.

Besides, it takes little effort to check from expression (11) that

Proposition 6. *$a : b ::_L c : d = 0$ if and only if*

- *$a - b = 1$ and $c \leq d$,*
- *or $b - a = 1$ and $d \leq c$,*
- *or $a \leq b$ and $c - d = 1$,*
- *or $b \leq a$ and $d - c = 1$*

Thus, $a : b ::_L c : d = 0$ when the change inside one of the pairs (a, b) or (c, d) is maximal, while the other pair shows either no change or any change in the opposite direction. Here, again there are more cases where the analogical proportion totally fails than in the previous subsection.

In the 3-valued case means the 22 following false patterns:

$$(1,1,1,0);\ (1,1,0,1);\ (1,0,1,1);\ (0,1,1,1);\ (0,0,0,1);\ (0,0,1,0);\ (0,1,0,0);$$
$$(1,0,0,0);\ (1,0,0,1);\ (0,1,1,0);\ (1,0,0,1/2);\ (0,1,1,1/2);\ (1,0,1/2,1);$$
$$(1/2,0,0,1);\ (0,1/2,1,0);\ (1,1/2,0,1);\ (0,1,1/2,0);\ (1/2,1,1,0),$$
$$(1,0,1/2,1/2);\ (0,1,1/2,1/2);\ (1/2,1/2,1,0);\ (1/2,1/2,0,1).$$

Note that the conservative approach (7) yields truth-value 1/2 (rather than 0) for the last 4 listed patterns. Thus, $a : b ::_L c : d = 1/2$ for $81 - 19 - 22 = 40$ distinct patterns when we use $\mathcal{L} = \{0, 1/2, 1\}$.

The discrepancy between (7) and (11) may be still better seen when considering so-called "continuous" proportions of the form $a : b :: b : c$. Indeed,

- with (7), $a : b ::_C b : c = 1 - \max(|a - b|, |b - c|)$, and thus $a : b :: b : c = 1$ if and only if $a = b = c$,
- while $a : b ::_L b : c = 1$ if and only if $(a - b) = (b - c)$, i.e., $b = (a + c)/2$.

This result suggests that the liberal approach (11) is suitable for interpolation and extrapolation purposes [19,20].

In particular, the following result is noticeable:

Proposition 7. *If A, C are the coordinates of points in a n-dimensional space, i.e. $A = (a_1, \cdots, a_n)$, $C = (c_1, \cdots, c_n)$, the solution of the analogical proportion equation*

$$A : X ::_L X : C = 1$$

is the midpoint $B = ((a_1 + c_1)/2, \cdots, (a_n + c_n)/2)$ of the segment AB in the n-dimensional space.

Proof. For each component, the equation reads: $x_i - a_i = c_i - x_i$, which yields $x_i = (a_i + c_i)/2$. $\square$

The solution of equation $A : X ::_L X : C = 1$ is thus obtained by a linear interpolation.

Similarly, the solution of equation $A : B ::_L C : X = 1$ is obtained by a linear extrapolation $x_i = c_i + b_i - a_i, \forall i$. This solution exists provided that $0 \le c_i + b_i - a_i \le 1$ for all $i = 1, \cdots, n$.

Remark 6. The same equations with the conservative approach generally have no solutions. Namely, $A : X ::_C X : C = 1$ only if $X = A = C$, for otherwise, generally, neither $A : B ::_C B : C$ nor $A : A ::_C A : C$, nor $A : C ::_C C : C$ can get value 1. Likewise $A : B ::_C C : X \neq 1$ if A, B, C take distinct values. Note that in the Boolean case, the extrapolation equation $A : B :: C : X = 1$ is solvable iff $(A \equiv B) \vee (A \equiv C)$ holds componentwise. In that case, the unique solution is $X = A \equiv (B \equiv C)$[11,12,16]. This solution can be adapted for solving $A : B ::_C C : X = 1$ but as pointed out in Proposition 2, each 4-tuple (a_i, b_i, c_i, d_i) must contain at most two distinct values, which comes down to very few feasible cases if values lie in [0, 1], whose practical interest looks limited.

5. Concluding remarks

The ambition of this paper is modest. It is an invitation to study, in the setting of fuzzy set theory, the notion of logical proportion pervading analogical reasoning. The multiple-valued extensions of various Boolean expressions of the logical proportions discussed here are some significant examples among other possible ones. In particular, a systematic investigation of algebraic structures needed to make sense of each expression of the logical proportion, especially, ordinal structures using residuated lattices with or without involutive negation, and the expressive power of the corresponding analogical reasoning patterns is a matter of future research.

In practice, the analogical inference machinery is based on the idea that if the same logical proportion holds for a number of components of A, B, C, D, then it may still hold for a new component known for A, B, C, but not for D, which can then be extrapolated. This has been applied to classification. The core principle of the algorithm is then to extrapolate the class of D on the basis of the best triple of examples A, B, C, such that $a_i : b_i :: c_i : d_i$ holds with a high degree for a maximum number of components (the best triple using a leximax ordering of the n-tuples of scores); see e.g., [18]. Although the multiple-valued extension of the analogical proportion in the liberal form of Subsection 4.2 may sound more appropriate for completing incomplete databases via extrapolation, the conservative extension discussed in this paper may be also of interest in classification. Judging the appropriateness of a particular extension for a given context still remains an open question.

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